Nonlinear PDEs

Smoluchowski eq.

Nonlinear PDEs

Drift-diffusion, with self-consistent drift.

Unknown : $(t,x) \longmapsto \rho(t,x) \geq 0$ space density

time variable, space variable

$$\partial_t \rho + \nabla_x \cdot (\rho u) = \Delta_x \rho$$

u = velocity field.

u derives from a potential

$$u = -\nabla_x \phi$$

which itself depends on the density through the Poisson eq

$$-\Delta_x \phi = \pm \rho .$$

application:

- sign $\oplus$: REPULSIVE force
  electrostatic potential
  ρ = density of electric charge
- sign $\ominus$ ATTRACTIVE force
  gravitational potential

The gravitational case is more difficult.

It arises:

- in astrophysics : ρ = density of stars

cf.: Smoluchowsky eq., see Chandrasekhar

- in biology : ρ = density of individuals (bacteria)

Keller-Segel-Patlak eq.

It describe CHEMOTACTIC dynamics:

the individuals react to a signal they are emitting themselves, and they are attracted in the direction of increasing signals.

If ϕ is given, we have

$$\partial_t \rho - \nabla\cdot\left(\rho \nabla \phi + \nabla \rho\right) = 0$$

Equilibrium sol:

$$\rho_{eq} \nabla \phi = - \nabla \rho_{eq}$$

$$\rho_{eq} = Z \exp(-\phi)$$

(Z = normalizing constant)

The eq. recasts as

$$\partial_t \rho - \nabla\cdot\left(\rho_{eq} \nabla \rho/\rho_{eq}\right) = 0$$

since

$$\rho_{eq} \nabla \rho/\rho_{eq} = \frac{\rho_{eq}}{\rho_{eq}^2}\left(\rho_{eq} \nabla \rho - \rho \underbrace{\nabla \rho_{eq}}_{= -\rho_{eq} \nabla \phi}\right)$$

Idea:

ρ relaxes towards $Z e^{-\phi}$.

Expanding formally we get:

$$\partial_t \rho - \nabla_x \phi \cdot \nabla_x \rho = \Delta_x \rho + \rho \underbrace{\Delta_x \phi}_{= \pm \rho}$$

$$\partial_t \rho - \nabla_x \phi \cdot \nabla_x \rho = \Delta_x \rho \pm \rho^2$$

Competition between the regularizing effect of the heating, and the explosive dynamics of the ODE $y' = y^2$ (attractive).

Diffusionless case and connection to Burgers eq:

in 1D:

$$\begin{cases} \partial_t \rho + u \partial_x \rho = -\rho \partial_x u = \rho^2 \\ u = -\partial_x \phi \\ \partial^2_{xx} \phi = -\partial_x u = \rho \end{cases}$$

It can be cast as:

$$-\partial_x \big[\underbrace{\partial_t u + u \partial_x u}_{\partial_t u + \partial_x (u^2/2)} \big] = 0$$

$$\partial_t u + \partial_x (u^2/2) = 0$$

Multi-dimensional case.

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ u = -\nabla\phi, \quad \Delta\phi = \rho \end{cases}$$

$\partial_t \rho - \nabla\phi\cdot\nabla\rho = \rho\Delta\phi = \rho^2$ idem $\begin{cases} \frac{d}{dt}\left[\rho(t,X)\right] = \rho^2(t,X) \\ \dot{X} = -\nabla\phi(t,X). \end{cases}$

The eq. cont. as $\nabla\cdot u = -\rho$

$$-\partial_t \nabla\cdot u + \nabla\cdot\left(-\nabla\cdot u\, u\right) = 0$$

$$= -\nabla\cdot\left[\partial_t u + u\,\nabla\cdot u\right] = 0$$

2D $\nabla\cdot V = \partial_1 V_1 + \partial_2 V_2$, $\nabla\cdot V = 0 \iff V = \nabla^\perp p = \begin{pmatrix} -\partial_2 p \\ \partial_1 p \end{pmatrix}$

$$\begin{cases} \partial_t u + u\,\nabla\cdot u + \nabla^\perp p = 0 \\ \partial_t \rho + \nabla\cdot(\rho u) = 0, \quad u = -\nabla\phi, \quad \Delta\phi = \rho \end{cases}$$

Adhesion dynamics - cf. F. Poupaud.

Derivation from VPFP $\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}\left[\nabla_x\phi\cdot\nabla_v f + \nabla_v(vf + \nabla_v f)\right]$

$$\Delta\phi = \rho = \int f\,dv$$

High-field regime.

$\partial_t \rho + \nabla_x\cdot J = 0$, $J = \int v f\,dv$

$\nabla_v\cdot\left((v + \nabla_x\phi) f + \nabla_v f\right) = 0$

$f = \frac{\rho}{2\pi}\exp\left(-\frac{|v + \nabla\phi|^2}{2}\right)$

$J = -\rho\nabla_x\phi$, $\Delta_x\phi = \rho$

A simple argument shows that blow-up of solutions might occur in finite time: the total mass is a critical threshold

Starting observation:

mass is conserved

$$\frac{d}{dt}\int \rho\,dx = 0\ ; \qquad \int \rho\,dx = \int \rho_0\,dx = M_0.$$

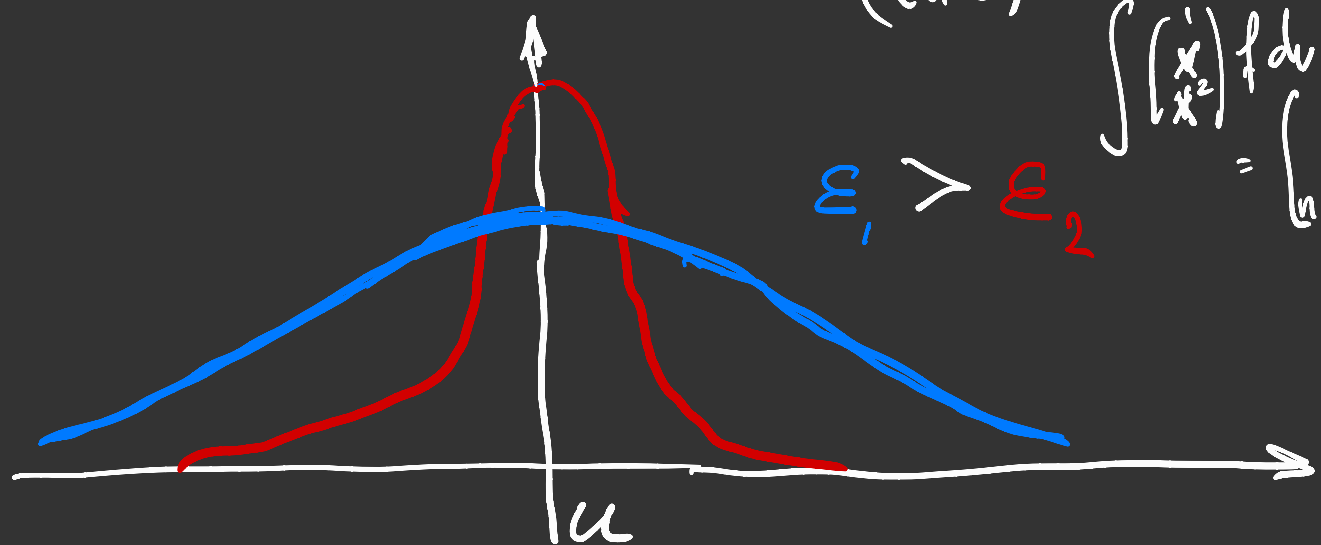
Compute the $2^{\underline{nd}}$ order moment

$$\int x^2 \rho(t,x)\,dx.$$

It measures the spreading of the sol.

(think of the gaussian $\dfrac{n}{(2\pi\theta)^{N/2}}\exp\left(-\dfrac{|v-u|^2}{2\theta}\right)$

$$\int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f\,dv = \begin{pmatrix} n \\ nu \\ nu^2 + Nn\theta \end{pmatrix}$$



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$$\frac{d}{dt}\int \frac{x^2}{2}\rho\, dx = \int \frac{x^2}{2}\nabla\cdot\left(\rho\nabla\phi + \nabla\rho\right) dx$$

$$= -\int x\cdot\left(\rho\nabla\phi + \nabla\rho\right) dx$$

Diffusion term :

$$\int x\cdot\nabla\rho\, dx = -\int \nabla\cdot x\ \rho\, dx = -N\int\rho = -N\Pi_0$$

Convection term : consider the case $\underline{\underline{N=2}}$

$$\phi(t,x) = \int \frac{\ln|x-y|}{2\pi}\rho(t,y)\, dy$$

$$\int x\rho\cdot\nabla\phi\, dx = \iint x\rho(x)\frac{x-y}{2\pi|x-y|^2}\rho(y)\, dy\, dx$$

$$= \frac{1}{4\pi}\iint \underbrace{\frac{x-y}{|x-y|^2}\cdot(x-y)}_{=\frac{|x-y|^2}{|x-y|^2} = 1}\rho(x)\rho(y)\, dy\, dx$$

$$= \frac{1}{4\pi}\int\rho(x)\, dx\int\rho(y)\, dy = \Pi_0^2/4\pi$$

We thus get

$$\frac{d}{dt}\int \frac{x^2}{2}\rho \, dx = 2M_0 \left(\underbrace{1}_{\text{diffusion}} - \underbrace{M_0/8\pi}_{\text{convection}}\right)$$

If $\frac{M_0}{8\pi} > 1$ then we obtain

$$\int \frac{x^2}{2}\rho(t,x)\, dx = \xi_0 - At$$

$$\xi_0 = \int \frac{x^2}{2}\rho_0 \, dx > 0, \qquad A = 2M_0\left(\frac{M_0}{8\pi} - 1\right)$$

... the 2^{nd} order moment would vanish and change sign at a critical time $T_0 = \xi_0/A$.

This is not consistent with the fact that ρ is a density, thus ≥ 0.

Therefore, the manipulation above are not permitted after T_0: a singularity occurs (and the integration by parts are not legitimate).

In higher dimensions, a similar computation leads to

$$\frac{d}{dt}\int \frac{x^2}{2}\rho\, dx = NM_0 - (N-2)C_N \iint x\cdot \frac{x-y}{|x-y|^N}\rho(x)\rho(y)\, dy\, dx$$

$$= NM_0 - \frac{(N-2)C_N}{2} \underbrace{\iint \rho(x)\rho(y) \frac{dx\,dy}{|x-y|^{N-2}}}_{\iint_{|x-y|>R} \cdots \;+\; \iint_{\underline{|x-y|<R}} \cdots}$$

$$\leq NM_0 - \frac{(N-2)C_N}{2R^{N-2}} \iint_{|x-y|<R} \rho(x)\rho(y)\, dy\, dx$$

$$\leq NM_0 - \frac{(N-2)C_N}{2R^{N-2}}\left(M_0^2 - \iint_{|x-y|>R} \rho(x)\rho(y)\, dy\, dx\right)$$

$$\frac{|x-y|}{R} > 1$$

$$\leq NM_0 - \frac{(N-2)C_N M_0^2}{2R^{N-2}} + \frac{(N-2)C_N M_0^2}{2R^N} \int_{/A} |x-y|^2 \rho(x)\rho(y)\, dx$$

$$2x^2 + 2y^2$$

We arrive at

$$\frac{d}{dt}\int \frac{x^2}{2}\rho\, dx \leq N M_0 - \frac{(N-2)C_N M^2}{2R^{N-2}} + \frac{2(N-2)C_N M}{R^N}\int x^2\rho\, dx$$

So far R is a free parameter. We set

$$R = \varepsilon\, M_0^{1/N-2}$$

and we obtain

$$\frac{d}{dt}\int \frac{x^2}{2}\rho\, dx \leq M_0 \overbrace{\left(N - \frac{C_N (N-2) M}{2\,\varepsilon^{N-2}}\right)}^{*}$$

$$+ \frac{2C_N (N-2) M^{1-N/N-2}}{\varepsilon^N}\int x^2\rho\, dx$$

We choose ε such that $* = -1$.

$$\frac{d}{dt}\int \frac{x^2}{2}\rho\, dx \leq M\left[K_{(N,M)}\int x^2\rho\, dx - 1\right]$$

This is a differential inequality

$$\mu' \leq M\left(K_{(N,M)}\,\mu - 1\right)$$

If at $t=0$, the RHS is <0, it remains <0 forever since μ is decreasing. Again this contradicts the global existence of solutions.

Rmk: blow-up does not occur in 1D with this argument since in 1D:

$$\frac{d}{dt}\int \frac{x^2}{2}\rho\,dx = M_0 + \iint x\,\rho(x)\rho(y)\,\frac{\mathrm{sgn}(x-y)}{2}\,dy\,dx$$

$$\leq M_0 + M_0 \int |x|\,\rho(x)\,dx$$

$$\leq M_0 \left(1 + 2\left(M_0 + \int x^2 \rho\,dx\right)\right)$$

and Grönwall's lemma implies a bound

$$\int \frac{x^2}{2}\rho\,dx \leq C_T \quad \text{for any } 0 \leq t \leq T < \infty$$

The case $N \geq 2$ with small data

$$\frac{d}{dt}\int \rho^p \, dx = + p \int \rho^{p-1} \nabla\cdot(\rho \nabla\phi + \nabla\rho)\, dx$$

$$= -p(p-1)\left\{ \int \rho^{p-2} |\nabla\rho|^2 \, dx + \int \rho^{p-2} \nabla\rho \, \rho \nabla\phi \, dx \right\}$$

Note that

$$|\nabla \rho^{p/2}|^2 = \left| \frac{p}{2} \rho^{\frac{p-2}{2}} \nabla\rho \right|^2 = \frac{p^2}{4} \rho^{p-2} |\nabla\rho|^2$$

$$= -4\,\frac{p-1}{p} \int |\nabla\rho^{p/2}|^2 \, dx + \underbrace{(p-1)\int \nabla\rho^p \cdot \nabla\phi \, dx}_{= -(p-1)\int \rho^p \Delta\phi \, dx = -(p-1)\int \rho^{p+1}}$$

That is:

$$\frac{d}{dt}\int \rho^p \, dx = -4\,\frac{p-1}{p}\int |\nabla \rho^{p/2}|^2 \, dx \boxed{+ (p-1)\int \rho^{p+1} \, dx}$$

Rmk: for repulsive forces, the RHS is ≤ 0 and we get favorable a priori estimates.

We can use the GAGLIARDO - NIRENBERG inequality

$$\begin{cases} \|u\|_{L^{\mathcal{P}}} \leq C \|\nabla u\|_{L^r}^{\theta} \|u\|_{L^q}^{1-\theta} \\ 1/\mathcal{P} = \theta\left(1/r - 1/N\right) + (1-\theta)/q \\ 0 \leq \theta \leq 1 \end{cases}$$

Dimension $N=2$

We apply this with $u = \rho^{p/2}$

$$|u|^{\mathcal{P}} = \rho^{p/2\,\mathcal{P}} = \rho^{p+1}$$

that is $\mathcal{P} = \frac{2p+2}{p} = 2\frac{p+1}{p}$

and $r = 2$, $q = 2/p$

$$1/\mathcal{P} = \frac{p}{2p+2} = \theta\left(1/2 - 1/2\right) + \frac{(1-\theta)}{2} p$$

yields $\theta = 1 - \frac{1}{p+1} = \frac{p}{p+1}$

and

$$\boxed{\int \rho^{p+1} dx} = \int |u|^{\mathcal{P}} dx \leq C \int |\nabla u|^2 dx \int |u|^{2/p} dx$$

$$\leq C \int |\nabla \rho^{p/2}|^2 dx \; \boxed{\underbrace{\int \rho \, dx}_{= M}}$$

So that

$$\frac{d}{dt} \int \rho^p dx \leq \boxed{\left(CM - \frac{4(p-1)}{p}\right)} \int |\nabla \rho^{p/2}|^2 dx$$

Hence $\int \rho^p dx$ is non increasing

if the initial mass is small enough.

Dimension $N>2$

We still work with $\mathfrak{p} = 2\frac{p+1}{p}$

and $u = \rho^{p/2}$ but now we have

$$1/\mathfrak{p} = \frac{p}{2(p+1)} = \theta\left(1/r - 1/N\right) + (1-\theta)/q$$

$$= \theta\left(1/2 - 1/N\right) + (1-\theta)/q$$

Select q such that $q\frac{p}{2} = N/2$, $q = N/p > 1$

We arrive at

$$\boxed{\int \rho^{p+1} dx} \leq C \int |\nabla \rho^{p/2}|^2 dx \left(\int \rho^{N/2} dx\right)^{2/N}$$

and finally

$$\frac{d}{dt}\int \rho^p dx \leq 4\frac{p-1}{p}\int |\nabla \rho^{p/2}|^2 dx \boxed{\left[C\frac{p}{4}\left(\int \rho^{N/2} dx\right)^{2/N} - 1\right]}$$

We use this relation with $p = N/2$:

if $\|\rho_0\|_{L^{N/2}}$ is small enough, then $\int \rho^{N/2} dx$

decays, $\iint |\nabla \rho^{N/4}|^2 dx$ is bounded, etc ...

This suggests to work with $p = N/2$
(which is thus meaningful for $N > 2$)
Setting $z(t) = \int \rho(t,x)^{N/2} dx$
we arrive at

$$\frac{d}{dt} z \leq C \underbrace{\left(K z^{2/N} - 1 \right)}_{*} \int |\nabla \rho^{N/2}|^2 dx$$

If initially $* < 0$, then z is decreasing
and $*$ remains < 0 forever.
These observations tell us that:
ρ is bounded in $L^\infty(0,T; L^{N/2}(\mathbb{R}^N))$
$\nabla \rho^{N/2}$ ——— $L^2((0,T) \times \mathbb{R}^N)$
$\rho^{N/2+1}$ is bounded in $L^1((0,T) \times \mathbb{R}^N)$
(by coming to the G.N. inequality)

The scheme of the proof is as follows:

- truncate/regularize the data
- replace $\phi = E * \rho$ by

$$\phi_\varepsilon = E * \zeta_\varepsilon * \rho$$

ζ_ε = mollifier $= \frac{1}{\varepsilon^N} \zeta(x/\varepsilon)$, $\zeta \in C_c^\infty \dots$

- prove existence-uniqueness of ρ_ε, sol. of the regularized pb.
- let $\varepsilon \to 0$ by using the estimate and compactness arguments (beware of time/space variables)

The system admits a "free energy" which is dissipated:

$$\frac{d}{dt}\int \Big(\underbrace{\rho \ln \rho}_{\text{entropy}} + \underbrace{\frac{\rho \phi}{2}}_{\text{potential energy}} \Big) dx$$

$$= \int \frac{\nabla \rho}{\rho} \left(\rho \nabla \phi + \nabla \rho \right) dx$$

$$- \underbrace{\frac{1}{2}\frac{d}{dt} \iint E(x-y)\, \rho(t,x)\, \rho(t,y)\, dx\, dy}$$ use symmetry

$$= \int \Big(\underbrace{\int E(x-y)\, \rho(t,y)\, dy}_{=\phi(t,x)} \Big) \nabla (\rho \nabla \phi + \nabla \rho)\, dx$$

$$= \int \left(\rho |\nabla \phi|^2 + \nabla \rho \nabla \phi \right)$$

$$= - \int \rho \, |\nabla \ln \rho + \nabla \phi|^2 \, dx.$$

Mind the sign of the potential: if ϕ is $\geqslant 0$ this provides a useful information: pb in the attractive case

Hardy-Littlewood-Sobolev inequality

If $f \geq 0$ lies in L^1 with $f \ln f \in L^1$, $\int f = M$, then

$$\int f \ln f \, dx + \frac{N}{M} \iint f(x) f(y) \ln |x-y| \, dy \, dx$$
$$\geq M \left(\ln M - C(N) \right).$$

In dimension $N=2$ it yields

$$\boxed{\left(1 - \frac{M}{8\pi}\right)} \int \rho \ln \rho \, dx \leq C^{st}.$$

It becomes useful with

$$\int \rho |\ln \rho| \, dx = \int \rho \ln \rho \, dx - 2 \int_{\rho \leq 1} \rho \ln \rho \, dx$$

$$= \int \rho \ln \rho \, dx - 2 \int_{\rho \leq e^{-x^2/2}} \rho \ln \rho \, dx - 2 \int_{e^{-x^2/2} \leq \rho \leq 1} \rho \ln \rho \, dx$$

$$\leq C^{st} + \int x^2 \rho \, dx + C \int e^{-x^2/4} \, dx \leq C_1.$$

It prevents blow-up formation!

if the mass is small enough.

Final touch: define the product $\rho \nabla \phi$.

We use symmetries so that

$$\int \rho \nabla \phi \; \nabla_2 \; dx$$

$$= \frac{1}{4\pi} \iint \rho(x)\rho(y) \underbrace{\frac{x-y}{|x-y|^2} \Big(\nabla_2(x) - \nabla_2(y) \Big)}_{\text{this lies in } L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} dy \, dx$$

A key argument: Simon's lemma

$X \Subset B \subset Y$ are Banach spaces

u_ε bounded in $L^p(0,T;X)$, $1 \leq p \leq \infty$

$\partial_t u_\varepsilon$ bounded in $L^1(0,T;Y)$

(or $L^r(0,T;Y)$ with $r>1$ if $p=+\infty$)

Then u_ε is relatively compact in $L^p(0,T;B)$

(or $C^0([0,T];B)$ if $p=+\infty$).

Ex: $X = H^1_0$, $Y = H^{-1}$, $B = L^2$

u_ε bounded in $L^2(0,T;H^1_0)$ | Ω bounded domain

$\partial_t u_\varepsilon$ ——— $L^2(0,T;H^{-1})$

then u_ε is relatively compact in $L^2((0,T)\times\Omega)$.

Lemma If $X \overset{c}{\subseteq} B \subset Y$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $x \in X$

$$\|x\|_B \leq \varepsilon \|x\|_X + C_\varepsilon \|x\|_Y$$

Proof. We argue by contradiction, assuming that for any $n \in \mathbb{N} \setminus \{0\}$, we can find $x_n \in X$ and $C > 0$ such that $\|x_n\|_B \geqslant C\|x_n\|_X + n\|x_n\|_Y$.

Wlog we can suppose $\|x_n\|_B = 1$

We deduce that:

$$\begin{cases} 1/C \geqslant \|x_n\|_X \\ 1/n \geqslant \|x_n\|_Y \end{cases}$$

Since $X \overset{c}{\subseteq} B$ and $(x_n)_{n\in\mathbb{N}}$ is bounded in X, we can suppose $x_n \xrightarrow[n\to+\infty]{} x$ in B.

Since $\|x_n\|_B = 1$, we get $\|x\|_B = 1$ which contradicts the fact that $x_n \xrightarrow[n\to+\infty]{} 0$ in Y.

For the Simon's lemma, we get

$$\|f_n(t+h) - f_n(t)\|_B \leq \varepsilon \|f_n(t+h) - f_n(t)\|_X + C_\varepsilon \|f_n(t+h) - f_n(t)\|_Y$$

<u>For</u> $p = +\infty$, we conclude directly by using Arzela-Ascoli's theorem:

$$\|f_n(t+h) - f_n(t)\|_B \leq 2C\varepsilon + C_\varepsilon \int_t^{t+h} \|\partial_t f_n(s)\|_Y \, ds$$

$$\leq 2C\varepsilon + \sup_n \|\partial_t f_n\|_{L^p(0,T;Y)} h^{1-1/p}$$

which is arbitrarily small, uniformly wrt n

<u>For $p=1$</u>, we have similarly

$$\int_0^{T-h} \|f_n(t+h) - f_n(t)\|_B \, dt \leq 2C\varepsilon + C_\varepsilon \int_0^{T-h}\int_0^h \|\partial_t f_n(s)\|_Y \, ds \, dt$$

$$\leq 2C\varepsilon + C_\varepsilon h$$

The case $1 < p < \infty$ can be treated by similar argument. We conclude by the Weyl-Kolmogorov-Fréchet argument

- $\displaystyle\int_{t_1}^{t_2} f(r)\,dr$ is compact in B for any t_1, t_2

- $\displaystyle\sup_{f \in \mathcal{F}} \int_0^{T-h} \| f(t+h) - f(r) \|_B^p \, dr \xrightarrow[h \to 0]{} 0$

Then $\mathcal{F}$ is compact in $L^p(0,T;B)$.

Proof. For $p = +\infty$, this is just Ascoli's theorem.

Let us set $\displaystyle T_a f(t) = \frac{1}{a} \int_t^{t+a} f(s)\,ds$

It satisfies

- $\displaystyle \| T_a f(t) \| \leq \frac{1}{a^{1/p}} \| f \|_{L^p(0,T;B)}$

- $\displaystyle \| T_a f(t+h) - T_a f(t) \|_B \leq \frac{1}{a} \left| \int_t^{t+h} \| f(s+h) - f(s) \|_B \, ds \right| \xrightarrow[h \to 0]{} 0$

uniformly on $\mathcal{F}$

- $\displaystyle \| T_a f - f \|_{L^p(0,T;B)} \leq \sup_{0 \leq s \leq a} \left(\int_0^{T-a} \| f(r+s) - f(r) \|_B^p \, dr \right)^{1/p}$

$\xrightarrow[a \to 0]{} 0$ uniformly on $\mathcal{F}$.